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Homological Methods for the study of C*-algebras(*)

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1. Introduction.

The main object of this expository note is to review very interesting recent developments of homological methods for the study of C*-algebras. Notable among these is the functor Ext (A) of Brown-Douglas-Fillmore [BDF], the K-functor for C*-algebras developed by Karoubi and others, and the Kasparov functor KK(A,B) [KAS]. These methods have been proved to be very sucessful for attacking various problems of C*-algebras and problems originated from operator theory and others.

It has also been recognized that these theories have deep relations with Atiyah-Singer index theory.

2. Extension functors.

Let X be a compact metric space, $C(X)$ the algebra of all continuous complex-valued functions on X. Let H be a separable infinite dimensional Hilbert space, $K(H)$ the ideal of all compact operators on H, $L(H)$ the algebra of all bounded linear operators on H. Then $K(H)$ is a norm closed two-sided ideal of $L(H)$ and hence one can form the quotient algebra $L(H)/K(H)$, which is also known as the Calkin algebra $Q(H)$. The algebras $C(X)$, $K(H)$, $L(H)$, $Q(H)$ are the most important examples of the so-called C*-algebras. By definition a C*-algebra A is an involutive Banach algebra with the C*-norm condition: $\|x^*x\|=\|x\|^2$ for all x in A. One of the main problem of C*-algebra theory is how to construct a new one from the old ones and how to classify those extended ones. Thus the immediate concrete question would be how to build new ones from $C(X)$ and $K(H)$ and how to classify them. One can restate the extension of $C(X)$ by $K(H)$ in terms of homogical algebra. Let

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$$O \longrightarrow K(H) \longrightarrow A \longrightarrow C(X) \longrightarrow O$$

be a short exact sequence. In this case morphisms are understood to be *-preserving algebra homomorphisms. It is easily seen that for each short exact sequence of the above we can associate a *-monomorphism τ:

$$\tau : C(X) \longrightarrow Q(H)$$

and vice versa. In this reason we call *-monomorphism $\tau : C(X) \to Q(H)$ an *extension* of $C(X)$. Two such extensions τ_1 and τ_2 are called *equivalent* if there exaists a unitary U in $L(H)$ such that $\tau_1(f)=\pi(U^*)\tau_2(f)\pi(U)$ for all f in $C(X)$, where π denotes the quotient map of $L(H)$ onto $Q(H)$. An extension τ is called *trivial* if there exists a representation σ of $C(X)$ into $L(H)$ such that $\tau=\pi\circ\sigma$.

Then all such trivial extensions are equivalent. Since there is a canonical way to embed $Q(H)\oplus Q(H)$ into $Q(H)$, we can define the sum $\tau_1+\tau_2$ of two extensions τ_1 and τ_2 by

$$(\tau_1+\tau_2)\ (f)= \begin{pmatrix} \tau_1(f) & 0 \\ 0 & \tau_2(f) \end{pmatrix} \in Q(H)\oplus Q(H) \longrightarrow Q(H)$$

Then in 1973 Brown of Purdue University, Douglas of SUNY at Stony Brook, and Fillmore of Dalhousie University proved the following beautiful theorem [BDF].

Theorem. Let X be a compact metric space. Then the equivalence classes $\mathrm{Ext}(X)$ of all extension is an abelian group.

We can propose the same problem for extensions of separable C*-algebras (not necessary commutative, notice that $C(X)$ is a commutative C*-algebra) by $K(H)$. Voiculescu proved that the equivalence classes of trivial extensions is the identity and later Choi and Effros provided the existence of the inverse. If X is a finite dimensional compact Hausdorff space and if A is a nuclear C*-algebra, then the so-called the homogeneous extensions of A by $C(X)\otimes K(H)$ (C*-tensor product of two algebras) is an abelian group $\mathrm{Ext}\ (X;A)$ [PPV], and it is a functor on two variables.

Let I be a closed two-sided ideal of A. Then we get a short exact sequence of C*-algebras

$$O \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow O$$

Kasparov's work [KAS] also shows that if we set

$$\mathrm{Ext}_1(A)=\mathrm{Ext}(A)$$

$$\mathrm{Ext}_0(A)=\mathrm{Ext}(C_0(R)\otimes A)$$

then there is a natural periodicity isomorphism theorem

$$\mathrm{Ext}(A) \cong \mathrm{Ext}(C_0(R^2) \otimes A)$$

Then we have the following theorem. This theorem is a special case of Kasparov's more general results.

Theorem. Let I be a closed two-sided ideal of a C*-algebra A. Then there a six-term exact sequence:

$$\begin{tikzcd} \mathrm{Ext}_1(A/I) \arrow[r] & \mathrm{Ext}_1(A) \arrow[r] & \mathrm{Ext}_1(I) \arrow[d] \\ \mathrm{Ext}_0(I) \arrow[u] & \mathrm{Ext}_0(A) \arrow[l] & Ext_0(A/I) \arrow[l] \end{tikzcd}$$

3. K-functors

Let $K^0(X)$ be the group generated by the isomorphism classes of vector bundles over a compact space X. Let S^nX to be the n-th reduced suspension of X. Let for any natural number n

$$K^{-n}(X) = K^0(S^nX).$$

Bott periodicity theorem says that $K^{-(n+2)}(X)$ and $K^{-n}(X)$ are isomorphic. Hence there are basically only two K-groups K^1 and K^0. And also $K^*(X)$ is a generalized cohomology theory. There are other ways to describe $K^*(X)$.

Namely, $K^0(X)$ is isomorphic to the formal differences of isomorphism classes of finitely generated projective $C(X)$-module. Or $K^0(X)$ can be seen as the group generated by the von Neumann equivalence classes of projections in the tensor product algebra $C(X) \otimes K(H)$. The last two descriptions of K-group lead us to the generalization of notions of K-group to the non-commutative C*-algebras. When A is a C*-algebra not necessary with identity, we let A^+ be the C*-algebra obtained by adjoining the identity to A. Let $U(A^+)$ denote the group consisting of all unitary elements in A^+ and $U_0(A^+)$ be the connected component of the identity in $U(\mathrm{A}^+)$. Then $U_0(A^+)$ is a normal subgroup of $U(A^+)$ and hence we can form the factor group $U(A^+)/U_0(A^+)$. It can be shown that $K^1(X)$ is isomorphic to the factor group $U((C(X) \otimes K(H))^+)/U_0((C(X) \otimes K(H))^+)$. These observation lead us to the following definitions.

Definition. Let A be a C*-algebra. The group generated by the von Neumann equvalence classes of projections in $A \otimes K(H)$ is called the K_0-***group***, $K_0(A)$, of A. The factor group

$U((A\otimes K(H)^{+}))/U_0((A\otimes K(H)^{+})$ is called the K_1-group, $K_1(A)$, of A.

Then we have the following powerful theorem.

Theorem Let I be a closed two-sided ideal of A. Then we have the following six-term exact sequence;

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{i_*} & K_0(A) & \xrightarrow{q_*} & K_0(A/I) \\ {\scriptstyle index}\uparrow & & & & \downarrow{\scriptstyle exp} \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

The horizontal maps i_* and q_* are map induced by the inclusion i; $I\rightarrow A$ and the quotient map q; $A\rightarrow A/I$, respectively. The map *index* is basically the map sending Fredholm elements in A/I to its index. And the other vertical map *exp* can be described as follows: Take projection p in A/I. We can find a self-adjoint element h in A such that $q(h)=p$. It is easily seen that the element $\exp(2\pi ih)$ is a unitary element in I^+. We can see that $exp(p)$ is the equivalence class in $K_1(I)$ generated by $\exp(2\pi ih)$. This is the basic reason why the name *exp* is given to this map.

Basically $K_0(A)$ tells us the structure of projections in A and $K_1(A)$ tells us the structure of unitary elements in A. This basic philosophy enables L.G. Brown to prove the outstanding question of AF-algebras. A C^*-algebra A is called an AF algebra (approximately finite algebra) if it is an inductive limit of increasing sequence of finite dimensional C^*-algebras. G. Elliott raised the following question: If I is a closed two -sided ideal of a C^*-algebra A and if both I and A/I are AF-algebras, then is A necessary an AF algebra? He recognized that this problem can be reduced to the question of whether one can lift any projection in the qustient algebra A/I to a projection in the original algebra A. For any AF algebras, its K_1-group is always the trivial zero group. Hence in the above six-term long exact sequence, bottom three terms are zero. Therefore i_* is an injection and q_* is a surjection. With these facts in hands, L.G. Brown was able to prove that any projection in the quotient algebra A/I can be lifted to a projection in the original algebra A.

Undoubtably the most important theorem in the K-theory is the Bott periodicity theorem. It has a long history of proofs. J. Mingo [M] unifies and clarifies Atiyah's techniques to the more general context. In his study of index theorem in terms of dynamical system of C^*-algebras $(A, \boldsymbol{R}, \alpha)$ and foliation theory, Connes [CON] proved the Thom isomorphism theorem (i.e., $K_i(A)$ and $K_{i+1}(A \underset{\alpha}{\times} \boldsymbol{R})$ are isomorphic). Very recently Paschke [P]

found a very simple proof of the Thom isomorphism theorem for a mapping torus of C^*-algebras.

4. Kasparov's KK-functor

There have been some outstanding partial relations between extension theory and K-theory. For an instance, $\mathrm{Ext}_1(X)$ and $\mathrm{Hom}(K^1(X), \boldsymbol{Z})$ are isomorphic if X is a compact set of complex plane. Also Paschke found that if A is a separable nuclear C^*-algebra and if $\pi(A)^c$ the commutant of $\pi(A)$ in the Calkin algebra then $K_0\ (\pi(A)^c)$ and $\mathrm{Ext}_1(A)$ are isomorphic. Recently Russian mathematician named Kasparov found a powerful and unifying theory which connects the extension theory on the one end and K-theory on the other end [KAS]. His proofs and methods are very hard to follow but in 1982 Cuntz of the University of Pennsylvania succeed to explain this theory in more natural context of C^*-algebra theory. The following definition is due to Cuntz [CUN].

Definition Let A and B be C^*-algebras. A *quasihomomorphism* from A to B is a pair $(\phi, \bar{\phi})$ of homomorphisms from A to E, where E is a C^*-algebra containing a subalgebra J of B as an ideal, such that $\phi(x)-\bar{\phi}(x)\in J$ for all $x\varepsilon A$.

We write $KK(A, B)$ for the set of all homotopy classes of quasihomomorphisms from A to $K(H)\otimes B$.

The following theorem is due to Kasparov.

Theorem Let A and B be C^*-algebras. Then $KK(A, B)$ is an abelian group. If $\boldsymbol{C}$ denotes the complex field, then we have.

$$KK(A, C)=\mathrm{Ext}_1(A),\quad KK(C, B)=K_0(B).$$

Kasparov's group $KK(A, B)$ is a covariant functor with respect to the first variables and a contravariant functor with respect to the second variables. Cuntz was able to prove the outstanding conjecture of Kadison: The reduced C^*-algebra of free group generated by two element contains no non-trivial projection.

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〈국문초록〉

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조 승 제

본 논문에서는 최근 급속히 발전되고 있는 C^*-대수에서의 비가환대수위상적인 방법들을 비교 검토하였으며 몇 개의 새로운 관점을 제시하였다. C^*-대수의 functor들은 Brown, Douglas, Fillmore의 확장이론, Ext(X), Karoubi 등에 의하여 발전된 C^*-대수들의 K-이론 그리고 Kasparov의 KK-이론이 대표적이다. 이들로써 설명되는 C^*-대수의 난해한 문제에 관하여도 고찰하였다.